Damascus University
Higher Institute of Earthquake Studies and Researches

Continuum Mechanics- Elasticity and Plasticity

Lec.07 Linear Stress-Strain Relation Problems

Problem 1:

تعطى الحالة الإجهادية في نقطة في وسط مستمر بالمركبات المعرفة من خلال تنسور كوتشي للإجهادات كما يلي:

1- أحسب مركبات تنسور الإجهادات σ في نظام جديد x'_1, x'_2, x'_3 .

2- أحسب اللامتغيرات الأساسية للإجهاد σ.

3- أحسب الإجهادات الرئيسية والإتجاهات الرئيسية.

$$\sigma_{ij}=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{pa}$$

$$a_{ij}=\frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

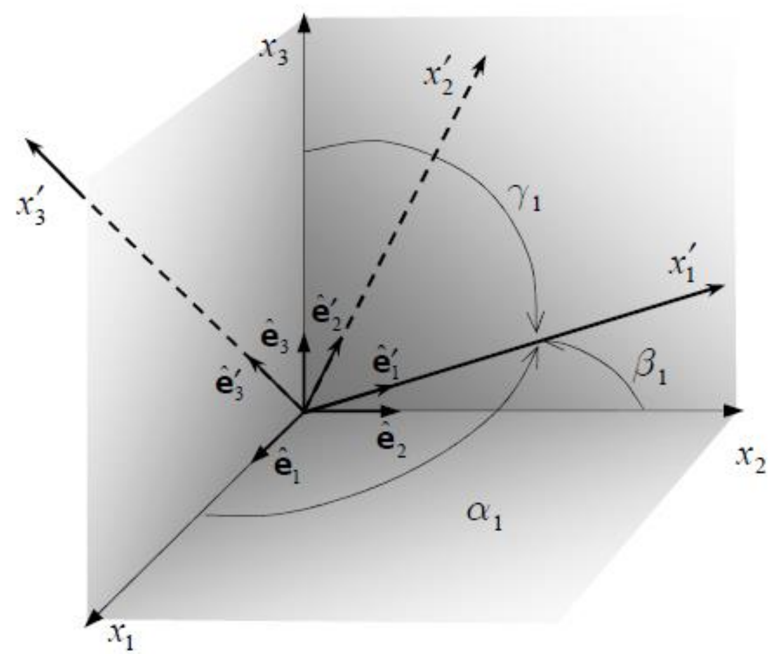
where

$$a_{11} = \cos\alpha_1$$

$$a_{12} = \cos\beta_1$$

$$a_{13} = \cos\gamma_1$$

$$\vdots$$

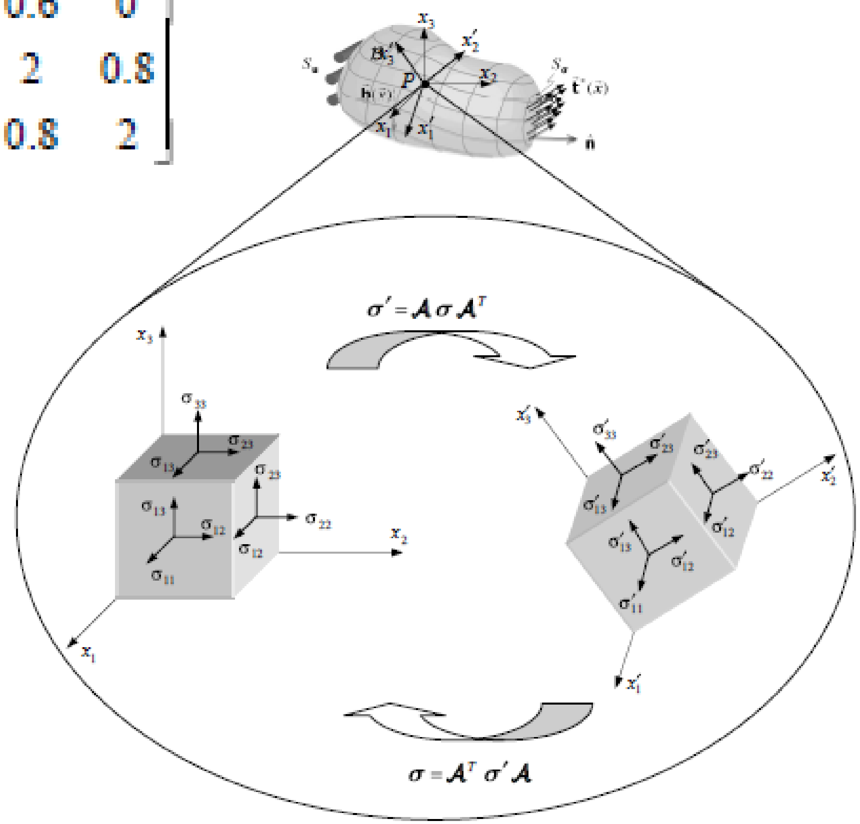


Problem 1:
Solution:

1- the transformation law for the components of a second order tensor is given by:

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl} \xrightarrow{\text{Matrix form}} \boldsymbol{\sigma}' = \mathcal{A}\,\boldsymbol{\sigma}\,\mathcal{A}^T$$

$$\sigma'_{ij} = \frac{1}{5^2}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0.6 & 0 \\ 0.6 & 2 & 0.8 \\ 0 & 0.8 & 2 \end{bmatrix}$$



Problem 1:
Solution:

2- The principal invariants of the Cauchy stress tensor can be calculated as follows: law for the components of a second order tensor is given by:

$$I_{\boldsymbol{\sigma}} = \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\mathit{II}_{\boldsymbol{\sigma}} = \frac{1}{2}\left[(\mathrm{Tr}\boldsymbol{\sigma})^2 - \mathrm{Tr}(\boldsymbol{\sigma}^2)\right] = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

$$= \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{33}\sigma_{22} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$\mathit{III}_{\boldsymbol{\sigma}} = \det(\boldsymbol{\sigma}) = \epsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} = \frac{1}{6}\left(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ii}\sigma_{jk}\sigma_{jk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}\right)$$

$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

By substituting the values of σ_{ij} for those in the proposed problem we obtain:

$$I_{\boldsymbol{\sigma}} = 6 \quad ; \quad \mathit{II}_{\boldsymbol{\sigma}} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 11 \quad ; \quad \mathit{III}_{\boldsymbol{\sigma}} = 6$$

Problem 1:
Solution:

c) The principal stresses (σ_i) and principal directions ($\hat{\mathbf{n}}^{(i)}$) are obtained by solving the following set of equations:

$$\begin{bmatrix} 2-\sigma & 1 & 0 \\ 1 & 2-\sigma & 0 \\ 0 & 0 & 2-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To obtain the nontrivial solutions of $\hat{\mathbf{n}}^{(i)}$ we have to solve the characteristic determinant, which is a cubic equation for the unknown magnitude σ:

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \quad \Rightarrow \quad \sigma^3 - I_\sigma \sigma^2 + II_\sigma \sigma - III_\sigma = 0$$

However, if we look at the format of the Cauchy stress tensor components, we can notice that we already have one solution as in the x_3-direction the tangential components are equal to zero, then:

$$\sigma_3 = 2 \xrightarrow{\text{Principal direction}} n_1^{(3)} = n_2^{(3)} = 0\,,\ n_3^{(3)} = 1$$

Problem 1:
Solution:

c) The principal stresses (σ_i) and principal directions $(\hat{\mathbf{n}}^{(i)})$ are obtained by solving the following set of equations:

To obtain the other two eigenvalues, one only need solve:

$$\begin{vmatrix} 2-\sigma & 1 \\ 1 & 2-\sigma \end{vmatrix} = (2-\sigma)^2 - 1 = 0 \quad \Rightarrow \quad \begin{cases} \sigma_1 = 1 \\ \sigma_2 = 3 \end{cases}$$

Then we can express the Cauchy stress tensor components in the principal space as:

$$\sigma'_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

Problem 1:
Solution:

c) The principal stresses (σ_i) and principal directions ($\hat{\mathbf{n}}^{(i)}$) are obtained by solving the following set of equations:

Additionally, the principal direction associated with $\sigma_1 = 1$ is calculated as follows:

$$\begin{bmatrix} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} n_1^{(1)} + n_2^{(1)} = 0 \\ n_1^{(1)} + n_2^{(1)} = 0 \end{cases} \Rightarrow n_1^{(1)} = -n_2^{(1)}$$

with $n_3^{(1)} = 0$ and by using the condition $n_1^{(1)2} + n_2^{(1)2} = 1$ we obtain:

$$n_1^{(1)} = -n_2^{(1)} = \frac{1}{\sqrt{2}} \text{ then } \hat{n}_i^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Since $\boldsymbol{\sigma}$ is a symmetric tensor, the principal space is formed by an orthogonal basis, so, it is valid that:

$$\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \quad ; \quad \hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \quad ; \quad \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(2)}$$

Problem 1:
Solution:

c) The principal stresses (σ_i) and principal directions ($\hat{\mathbf{n}}^{(i)}$) are obtained by solving the following set of equations:

Thus, the second principal direction can be obtained by the cross product between $\hat{\mathbf{n}}^{(3)}$ and $\hat{\mathbf{n}}^{(1)}$, *i.e.*:

$$\hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\hat{\mathbf{e}}_1 + \frac{1}{\sqrt{2}}\hat{\mathbf{e}}_2$$

which can also be checked by the following analysis:

The Principal direction associated with $\sigma_2 = 3$:

Problem 1:
Solution:

$$\begin{bmatrix} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \\ 0 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_1^{(2)} - n_2^{(2)} = 0 \end{cases} \Rightarrow n_1^{(2)} = n_2^{(2)}$$

With $n_3^{(3)} = 0$ and using the condition $n_1^{(3)2} + n_2^{(3)2} = 1$ we obtain:

$$n_1^{(2)} = n_2^{(2)} = \frac{1}{\sqrt{2}} \text{ then } \hat{n}_i^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

As we have seen in Chapter 1, the eigenvectors of a symmetric tensor form the transformation matrix $\mathcal{D}$, from the original system to the principal space, *i.e.* $\boldsymbol{\sigma}'' = \mathcal{D}\,\boldsymbol{\sigma}\,\mathcal{D}^T$, thus:

$$\begin{bmatrix} \sigma_1 = 1 & 0 & 0 \\ 0 & \sigma_2 = 3 & 0 \\ 0 & 0 & \sigma_3 = 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2:

At point P the Cauchy stress tensor components are:

$$\sigma_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} MPa$$

Find:

a) the traction vector $\vec{\mathbf{t}}$ related to the plane which is normal to the x_1-axis;

b) the traction vector $\vec{\mathbf{t}}$ associated with the plane whose normal is $(1,-1,2)$;

c) the traction vector $\vec{\mathbf{t}}$ associated with the plane parallel to the plane $2x_1 - 2x_2 - x_3 = 0$;

d) the principal stress at the point P;

e) the principal directions of $\boldsymbol{\sigma}$ at the point P.

Problem 2:

Solution:

a) In this case, the unit vector is $(1,0,0)$. Then, the traction vector is given by:

$$t_i^{(\hat{n})} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

b) The unit vector associated with the direction $(1,-1,2)$ is:

$$\hat{n}_i = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

thus,

$$t_i^{(\hat{n})} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 5 \\ 10 \\ -1 \end{bmatrix}$$

Problem 2:

Solution:

c)

$$\hat{n}_i = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \Rightarrow t_i^{(\hat{n})} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 \\ -10 \\ -7 \end{bmatrix}$$

d) Solving the characteristic determinant

$$\begin{vmatrix} 1-\sigma & 2 & 3 \\ 2 & 4-\sigma & 6 \\ 3 & 6 & 1-\sigma \end{vmatrix} = 0$$

we obtain:

$$\sigma_1 = 10\,;\ \sigma_2 = 0\,;\ \sigma_3 = -4$$

Problem 2:

Solution:

e) The principal directions are:

Associated with $\sigma_1 = 10$

$$\begin{cases} -9n_1 + 2n_2 + 3n_3 = 0 \\ 2n_1 - 6n_2 + 6n_3 = 0 \\ 3n_1 + 6n_2 - 9n_3 = 0 \end{cases} \Rightarrow n_i^{(1)} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$$

Similarly, we obtain:

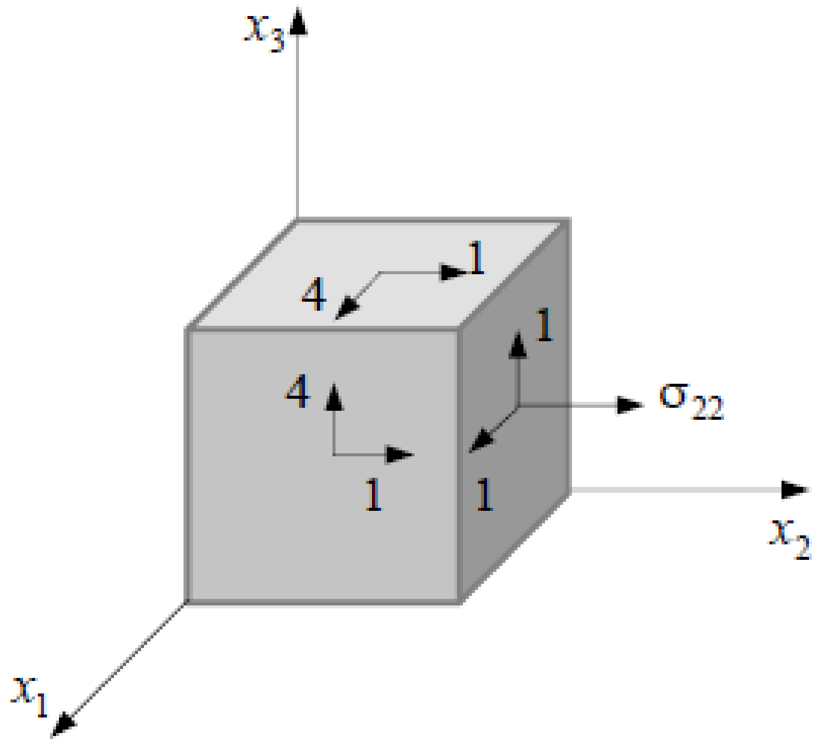
$$n_i^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \; n_i^{(3)} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Normalization of the principal directions:

$$\hat{n}_i^{(1)} = \frac{n_i^{(1)}}{\left\| \vec{n}^{(1)} \right\|} = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \quad ; \quad \hat{n}_i^{(2)} = \frac{n_i^{(2)}}{\left\| \vec{n}^{(2)} \right\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \hat{n}_i^{(3)} = \frac{n_i^{(3)}}{\left\| \vec{n}^{(3)} \right\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Problem 3:

The stress state at one point P of the continuous medium is given schematically by:



Obtain the value of the component σ_{22} of the Cauchy stress tensor such that there is at least one plane passing through P in which is free of stress;

Obtain the direction of the plane.

Problem 3:
Solution:

We seek to find a plane whose direction is $\hat{n}$ such that $\vec{t}^{(\hat{n})} = \vec{0}$. We can relate the Cauchy stress tensor to the traction vector by means of the equation:

$$\vec{t}^{(\hat{n})} = \boldsymbol{\sigma} \cdot \hat{n}$$

thus:

$$\begin{bmatrix} t_1^{(\hat{n})} \\ t_2^{(\hat{n})} \\ t_3^{(\hat{n})} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & \sigma_{22} & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Resulting in the following system of equations:

$$\begin{cases} n_2 + 4n_3 = 0 \Rightarrow n_3 = -\dfrac{1}{4}n_2 \\ n_1 + \sigma_{22}n_2 + n_3 = 0 \\ 4n_1 + n_2 = 0 \Rightarrow n_1 = -\dfrac{1}{4}n_2 \end{cases}$$

Problem 3:

Solution:

By combining the above equations we obtain:

$$n_1 + \sigma_{22} n_2 + n_3 = 0 \quad \Rightarrow \quad -\frac{1}{4} n_2 + \sigma_{22} n_2 - \frac{1}{4} n_2 = 0$$

$$\left(-\frac{1}{4} + \sigma_{22} - \frac{1}{4} \right) n_2 = 0$$

Then, for $\vec{n} \neq \vec{0}$, we have: $\left(-\frac{1}{4} + \sigma_{22} - \frac{1}{4} \right) = 0 \Rightarrow \sigma_{22} = \frac{1}{2}$.

To determine the direction of the plane we start by the restriction $n_i n_i = 1$, then:

$$n_i n_i = 1 \therefore \quad n_1^2 + n_2^2 + n_3^2 = 1$$

$$\Rightarrow \left(-\frac{1}{4} n_2 \right)^2 + n_2^2 + \left(-\frac{1}{4} n_2 \right)^2 = 1$$

$$\Rightarrow n_2 = \frac{2\sqrt{2}}{3} \quad ; \quad n_1 = n_3 = -\frac{\sqrt{2}}{6}$$

Thus, the direction of the normal to the plane, when it meets $\vec{t}^{(\hat{n})} = \vec{0}$:

$$\hat{n}_i = \frac{\sqrt{2}}{6} \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

Problem 4:

If the state of stress at a point in a body is given as follows. Determine the components of the body force in order to satisfy the equations of equilibrium.

$$\sigma_x = 20x^3 + y^2;\ \sigma_y = 30x^3 + 100;;\ \sigma_z = 10(y^2 + z^2)$$

&

$$\tau_{xy} = z;\ \tau_{yz} = x^3;\ \tau_{zx} = y^3$$

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Problem 5:

Ignoring the curvature of the Earth's surface, the gravitational field can be assumed to be uniform as shown in Figure 3.1, where g is the acceleration caused by gravity (the gravity of the Earth). Find the resultant force acting on the body $\mathcal{B}$.

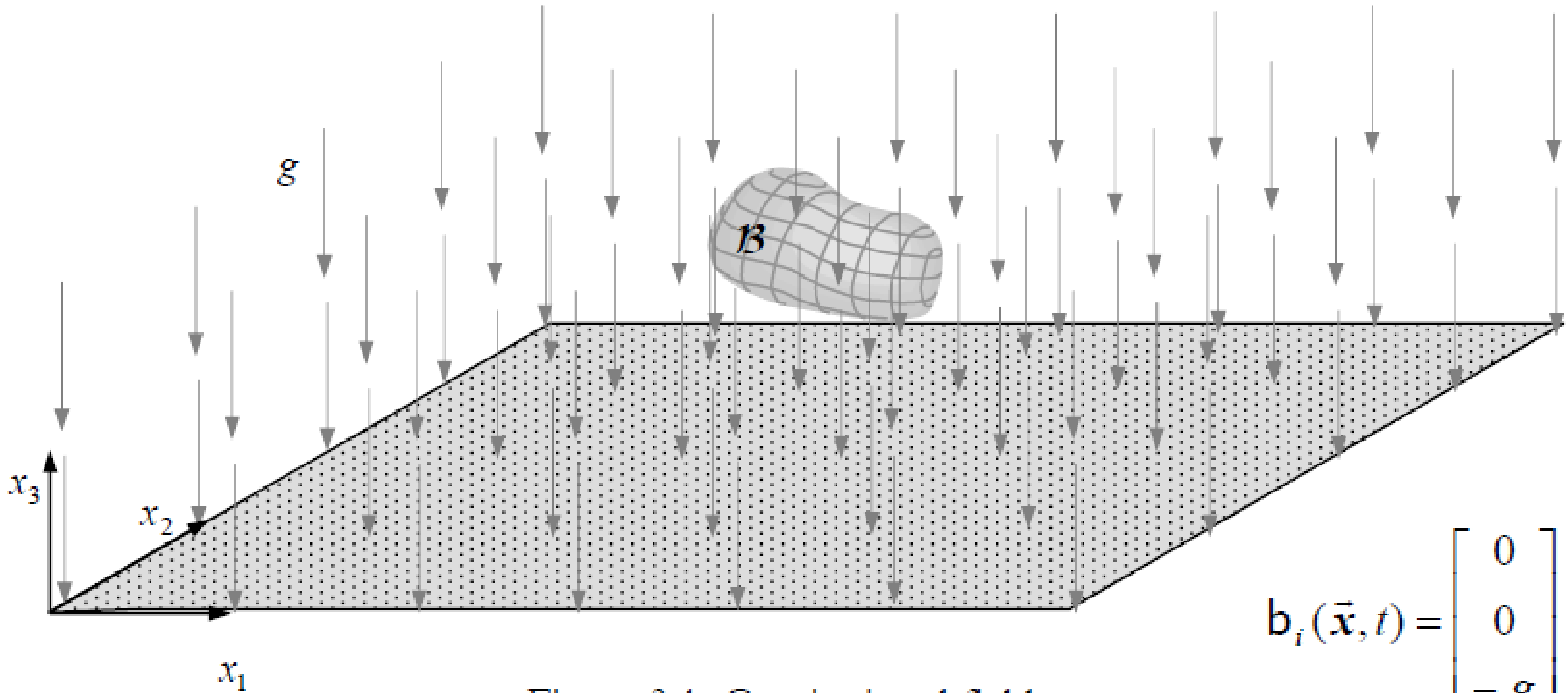

Figure 3.1: Gravitational field.

$$\mathsf{b}_i(\vec{x}, t) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \left[\frac{m}{s^2} \right]$$

All bodies immersed in a force field are subjected to the body force $\vec{\mathbf{b}}$

Problem 5:
solution:

All bodies immersed in a force field are subjected to the body force $\vec{\mathbf{b}}$

$$b_i(\vec{x},t) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \left[\frac{m}{s^2}\right]$$

Hence, the total force acting on the body can be evaluated as follows:

$$F_i = \int_V \rho b_i(\vec{x},t)\, dV = \begin{bmatrix} 0 \\ 0 \\ -\int_V \rho g \, dV \end{bmatrix}$$

We can also verify the F unit: $[F] = \int_V \left[\frac{kg}{m^3}\right]\left[\frac{m}{s^2}\right] \overbrace{dV}^{[m^3]} = \frac{kg\, m}{s^2} = N(Newton)$.

Problem 6:

Given the tensor components:

$$\mathsf{T}_{ij} = \begin{bmatrix} 5 & 3 & 3 \\ 2 & 6 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

a) Obtain the principal invariants of $\mathbf{T}$, *i.e.* obtain $I_{\mathbf{T}}$, $II_{\mathbf{T}}$ and $III_{\mathbf{T}}$;

b) Obtain the characteristic polynomial associated with $\mathbf{T}$;

c) If λ_1, λ_2 y λ_3 are the eigenvalues of $\mathbf{T}$ and $\lambda_1 = 10$. Obtain λ_2 and $\lambda_3 > 2$.

Problem 6:
Solution:

a) The principal invariants of $\mathbf{T}$ are:

$$I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = 5 + 6 + 4 = 15$$

$$II_{\mathbf{T}} = \begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} = 56$$

$$III_{\mathbf{T}} = \det(\mathbf{T}) = 60$$

b) The characteristic polynomial can be obtained by solving the determinant:

$$\begin{vmatrix} 5-\lambda & 3 & 3 \\ 2 & 6-\lambda & 3 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0$$

thus:

$$\lambda^3 - 15\lambda^2 + 56\lambda - 60 = 0$$

Problem 6:

Solution:

c) In the principal space the following is true:

$$\mathrm{T}'_{ij} = \begin{bmatrix} \lambda_1 = 10 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 > 2 \end{bmatrix}$$

where the principal invariants are

$$I_{\mathrm{T}} = \mathrm{Tr}(\mathbf{T}) = \lambda_1 + \lambda_2 + \lambda_3 = 15 \quad \Rightarrow \quad \lambda_2 + \lambda_3 = 5$$

$$I\!I\!I_{\mathrm{T}} = \det(\mathbf{T}) = \lambda_1 \lambda_2 \lambda_3 = 60 \quad \Rightarrow \quad \lambda_2 \lambda_3 = 6$$

By combining these two equations we obtain:

$$\left.\begin{aligned} \lambda_2 \lambda_3 &= 6 \\ \lambda_2 + \lambda_3 &= 5 \end{aligned}\right\} \Rightarrow (5 - \lambda_3)\lambda_3 = 6 \Rightarrow \lambda_3^2 - 5\lambda_3 + 6 = 0 \Rightarrow \begin{cases} \lambda_3^{(1)} = 3 \\ \lambda_3^{(2)} = 2 \end{cases}$$

We discard the solution $\lambda_3^{(2)} = 2$, thus $\lambda_3 = 3$. Then:

$$\mathrm{T}'_{ij} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 6:

Solution:

In this space we can check that
$$\begin{cases} I_{\mathrm{T}} = \mathrm{Tr}(\mathbf{T}) = 10 + 2 + 3 = 15 \\ II_{\mathrm{T}} = 2\times 3 + 10\times 3 + 10\times 2 = 56 \\ III_{\mathrm{T}} = \det(\mathbf{T}) = 10\times 2\times 3 = 60 \end{cases}$$

Problem 7:

Find the principal values and directions of the second-order tensor $\mathbf{T}$, where the Cartesian components of $\mathbf{T}$ are:

$$(\mathbf{T})_{ij} = \mathrm{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 7:
Solution:

Solution: We need to find nontrivial solutions for $(T_{ij} - \lambda\delta_{ij})\hat{n}_j = 0_i$, which are constrained by $\hat{n}_j\hat{n}_j = 1$ (unit vector). As we have seen, the nontrivial solution requires that:

$$\left| T_{ij} - \lambda\delta_{ij} \right| = 0$$

Explicitly, the above equation is:

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Developing the above determinant, we can obtain the cubic equation:

$$(1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0$$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

We could have obtained the characteristic equation directly in terms of invariants:

$$I_T = \mathrm{Tr}(T_{ij}) = T_{ii} = T_{11} + T_{22} + T_{33} = 7$$

$$II_T = \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ij}\right) = \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = 14$$

$$III_T = \left| T_{ij} \right| = \epsilon_{ijk} T_{i1} T_{j2} T_{k3} = 8$$

Problem 7:
Solution:

Then, the characteristic equation becomes:

$$\lambda^3 - \lambda^2 I_T + \lambda II_T - III_T = 0 \quad \rightarrow \quad \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

On solving the cubic equation we obtain three real roots, namely:

$$\lambda_1 = 1; \quad \lambda_2 = 2; \quad \lambda_3 = 4$$

We can also verify that:

$$I_T = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 4 = 7 \quad \checkmark$$

$$II_T = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\times 2 + 2\times 4 + 4\times 1 = 14 \quad \checkmark$$

$$III_T = \lambda_1\lambda_2\lambda_3 = 8 \quad \checkmark$$

Thus, we can see that the invariants are the same as those evaluated previously.

Problem 7:
Solution:

Principal directions:

Each eigenvalue, λ_i, is associated with a corresponding eigenvector, $\hat{\mathbf{n}}^{(i)}$. We can use the equation $(T_{ij} - \lambda\delta_{ij})\hat{n}_j = 0_i$ to obtain the principal directions.

• $\lambda_1 = 1$

$$\begin{bmatrix} 3-\lambda_1 & -1 & 0 \\ -1 & 3-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-1 & -1 & 0 \\ -1 & 3-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These become the following system of equations:

$$\begin{cases} \left.\begin{aligned} 2n_1 - n_2 = 0 \\ -n_1 + 2n_2 = 0 \end{aligned}\right\} \Rightarrow n_1 = n_2 = 0 \\ 0n_3 = 0 \end{cases}$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1$$

Then we can conclude that: $\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$.

Problem 7:
Solution:

NOTE: This solution could have been directly determined by the specific features of the $\boldsymbol{T}$ matrix. As the terms $T_{13} = T_{23} = T_{31} = T_{32} = 0$ imply that $T_{33} = 1$ is already a principal value, then, consequently, the original direction is a principal direction. ■

$\lambda_2 = 2$

$$\begin{bmatrix} 3-\lambda_2 & -1 & 0 \\ -1 & 3-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \\ 0 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} n_1 - n_2 = 0 \Rightarrow n_1 = n_2 \\ -n_1 + n_2 = 0 \\ -n_3 = 0 \end{cases}$$

The first two equations are linearly dependent, after which we need an additional equation:

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_1^2 = 1 \Rightarrow n_1 = \pm\sqrt{\frac{1}{2}}$$

Thus:

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \begin{bmatrix} \pm\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

Problem 7:
Solution:

$\lambda_3 = 4$

$$\begin{bmatrix} 3-\lambda_3 & -1 & 0 \\ -1 & 3-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \\ 0 & 0 & 1-4 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} \left.\begin{aligned} -n_1 - n_2 = 0 \\ -n_1 - n_2 = 0 \end{aligned}\right\} \Rightarrow n_1 = -n_2 \\ -3n_3 = 0 \end{cases}$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_2^2 = 1 \Rightarrow n_2 = \pm\sqrt{\frac{1}{2}}$$

Then:

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \begin{bmatrix} \mp\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

Afterwards, we summarize the eigenvalues and eigenvectors of T :

Problem 7:
Solution:

$$\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = \begin{bmatrix} 0 & 0 & \pm 1 \end{bmatrix}$$

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \begin{bmatrix} \pm\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \begin{bmatrix} \mp\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

NOTE: The tensor components of this problem are the same as those used in **Problem 1.62**. Additionally, we can verify that the eigenvectors make up the transformation matrix, $\mathcal{A}$, between the original system, (x_1, x_2, x_3), and the principal space, (x_1', x_2', x_3'), (see **Problem 1.62**). ■

Problem 8:

Given a body in equilibrium in which the Cauchy stress tensor field is represented by its components:

$$\sigma_{11} = 6x_1^3 + x_2^2 \quad ; \quad \sigma_{12} = x_3^2$$
$$\sigma_{22} = 12x_1^3 + 60 \quad ; \quad \sigma_{23} = x_2$$
$$\sigma_{33} = 18x_2^3 + 6x_3^3 \quad ; \quad \sigma_{31} = x_1^2$$

Obtain the body force vector (per unit volume) at the point $(x_1 = 2; x_2 = 4; x_3 = 2)$.

Problem 8:
Solution:

The equilibrium equations:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \vec{0}$$

$$\begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} + \dfrac{\partial\sigma_{13}}{\partial x_3} + \rho b_1 = 0 \Rightarrow \rho b_1 = -\dfrac{\partial\sigma_{11}}{\partial x_1} - \dfrac{\partial\sigma_{12}}{\partial x_2} - \dfrac{\partial\sigma_{13}}{\partial x_3} \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} + \rho b_2 = 0 \Rightarrow \rho b_2 = -\dfrac{\partial\sigma_{21}}{\partial x_1} - \dfrac{\partial\sigma_{22}}{\partial x_2} - \dfrac{\partial\sigma_{23}}{\partial x_3} \\ \dfrac{\partial\sigma_{31}}{\partial x_1} + \dfrac{\partial\sigma_{32}}{\partial x_2} + \dfrac{\partial\sigma_{33}}{\partial x_3} + \rho b_3 = 0 \Rightarrow \rho b_3 = -\dfrac{\partial\sigma_{31}}{\partial x_1} - \dfrac{\partial\sigma_{32}}{\partial x_2} - \dfrac{\partial\sigma_{33}}{\partial x_3} \end{cases}$$

$$\begin{cases} \rho b_1 = -18x_1^2 - 0 - 0 \\ \rho b_2 = -0 - 0 - 0 \\ \rho b_3 = -2x_1 - 1 - 18x_2^2 \end{cases} \quad \Rightarrow \quad \rho b_i = \begin{bmatrix} -18x_1^2 \\ 0 \\ -2x_1 - 1 - 18x_2^2 \end{bmatrix}$$

At the point $x_1 = 2; x_2 = 4; x_3 = 2$ we obtain:

$$\rho b_i = \begin{bmatrix} -72 \\ 0 \\ -77 \end{bmatrix}$$

(Force per unit volume)